The relation of semiadjacency in transformative $\cap$-semigroups

W. A. Dudek and V. S. Trokhimenko

Abstract

We consider semigroups of transformations (partial mappings defined on a set A) closed under the set-theoretic intersection of mappings treated as subsets of $A\times A$. On such semigroups we define two relations: the relation of semicompatibility which identifies two transformations at the intersection of their domains and the relation of semiadjacency when the image of one transformation is contained in the domain of the second. Abstract characterizations of such semigroups are presented.

Keywords: Semigroup; Semigroup of transformations; Algebra of functions.

1. Let $\mathcal{F}(A)$ be the set of all transformations (i.e., the partial maps) of a non-empty set A. The composition (superposition) of such maps is defined as $(g\circ f)(a)=g(f(a))$, where for every $a\in A$ the left and right hand side are defined, or undefined, simultaneously (cf. [1]). If the set $\Phi\subset\mathcal{F}(A)$ is closed with respect to such composition, then the algebra $(\Phi,\circ)$ is called a *semigroup of transformations* (cf. [1] or [10]). If Φ is also closed with respect to the set-theoretic intersection of transformations treated as subsets of $A\times A$, then the algebra $(\Phi,\circ,\cap)$ is called a $\cap$-*semigroup of transformations*.

On such $\cap$-semigroup we can consider the so-called *semicompatibility relation* ξ_Φ defined as follows:

$$(f,g)\in\xi_\Phi\longleftrightarrow f\circ\Delta_{\mathrm{pr}_1 g}=g\circ\Delta_{\mathrm{pr}_1 f},\tag{1}$$

where $\mathrm{pr}_1 f$ is the domain of f and $\Delta_{\mathrm{pr}_1 f}$ is the identity relation on $\mathrm{pr}_1 f$. The algebraic system $(\Phi,\circ,\cap,\xi_\Phi)$ is called a *transformative* $\cap$-*semigroup of transformations*. The investigation of such semigroups was initiated by V. V. Vagner (cf. [14]) and continued by V. N. Saliĭ and B. M. Schein (cf. [7], [8] and [9]). The first abstract characterization of a $\cap$-semigroup of transformations was found by V. S. Garvatskiĭ (cf. [4]).

Some abstract characterizations of transformative $\cap$-semigroups of transformations can be deduced from results proved in [2] and [13] for Menger $\cap$-algebras of n-place functions.

On $(\Phi,\circ)$ we can also consider the *semiadjacency relation*

$$\delta_\Phi=\{(f,g)\,|\,\mathrm{pr}_2 f\subset\mathrm{pr}_1 g\},$$

where $\mathrm{pr}_2 f$ is the image of f.

The abstract characterization of semigroups of transformations with this relation was given in [6]. Later, in [5], was found an abstract characterization of semigroups of transformations containing these two relations, i.e., an abstract characterization of an algebraic system $(\Phi, \circ, \xi_\Phi, \delta_\Phi)$. A $\cap$-semigroup of transformations with the semiadjacency relation was described in [3]. The semiadjecency relation on algebras of multiplace functions was investigated in [11].

In this paper we find an abstract characterization of a $\cap$-semigroup of transformations containing the semicompatibility relation and the relation of semiadjacency.

We start with the following lemma.

Lemma 1. *The relation of semiadjacency defined on a semigroup* $(\Phi, \circ)$ *satisfies the following two conditions:*

$$(f, g) \in \delta_\Phi \longleftrightarrow pr_1 f \subset pr_1 (g \circ f), \tag{2}$$

$$(f, g) \in \delta_\Phi \longrightarrow (f \circ h, g) \in \delta_\Phi. \tag{3}$$

We omit the proof of this lemma since it is a simple consequence of results proved in [3], [5] and [6].

2. Each homomorphism P of an abstract semigroup $(G, \cdot)$ into a semigroup $(\mathcal{F}(A), \circ)$ of all transformations of a set A is called a *representation of* $(G, \cdot)$ *by transformations*. In the case when a representation is an isomorphism we say that it is *faithful*.

With each representation P of a semigroup $(G, \cdot)$ by transformations of A are associated three binary relations:

$$\zeta_P = \{(g_1, g_2) \mid P(g_1) \subset P(g_2)\},$$
$$\xi_P = \{(g_1, g_2) \mid P(g_1) \circ \triangle_{\mathrm{pr}_1 P(g_2)} = P(g_2) \circ \triangle_{\mathrm{pr}_1 P(g_1)}\},$$
$$\delta_P = \{(g_1, g_2) \mid \mathrm{pr}_2 P(g_1) \subset \mathrm{pr}_1 P(g_2)\}$$

defined on G.

Let $(P_i)_{i \in I}$ be the family of representations of a semigroup $(G, \cdot)$ by transformations of disjoint sets $(A_i)_{i \in I}$, respectively. By the *sum* of this family we mean the map $P\colon\ g \mapsto P(g)$, where $g \in G$, and $P(g)$ is a transformation on $A = \bigcup\limits_{i \in I} A_i$ defined by

$$P(g) = \bigcup_{i \in I} P_i(g).$$

Such defined P is a representation of $(G, \cdot)$. It is denoted by $\sum\limits_{i \in I} P_i$.

If P is a sum of family representations $(P_i)_{i \in I}$, then obviously

$$\zeta_P = \bigcap_{i \in I} \zeta_{P_i}, \quad \xi_P = \bigcap_{i \in I} \xi_{P_i}, \quad \delta_P = \bigcap_{i \in I} \delta_{P_i}. \tag{4}$$

3. Following [1] and [10] a binary relation ρ on a semigroup $(G,\cdot)$ is called:

- *stable* or *regular*, if for all $x,y,u,v\in G$

$$(x,y)\in\rho\wedge(u,v)\in\rho\longrightarrow(xu,yv)\in\rho,$$

- *left regular*, if for all $x,u,v\in G$

$$(u,v)\in\rho\longrightarrow(xu,xv)\in\rho,$$

- *right regular*, if for all $x,y,u\in G$

$$(x,y)\in\rho\longrightarrow(xu,yu)\in\rho,$$

- a *left ideal*, if for all $x,y,u\in G$

$$(x,y)\in\rho\longrightarrow(ux,y)\in\rho,$$

- *right negative*, if for all $x,y,u\in G$

$$(x,yu)\in\rho\longrightarrow(x,y)\in\rho.$$

A quasi-order ρ, i.e., a reflexive and transitive relation, is stable if and only if it is left and right regular (cf. [1], [10]). Similarly, it is right negative if $(xy,x)\in\rho$ for all $x,y\in G$.

By the *determining pair* of a semigroup $(G,\cdot)$ we mean an ordered pair (ε,W), where ε is a right regular equivalence relation on a semigroup $(G^*,\cdot)$, [1] and W is the empty set or an ε-class which is a right ideal of $(G,\cdot)$. Let $(H_a)_{a\in A}$ be a collection of all ε-classes (uniquely indexed by elements of A) such that $H_a\neq W$. As is well known (cf. [10]) with each determining pair (ε,W) is associated the so-called *simplest representation* $P_{(\varepsilon,W)}$ of $(G,\cdot)$ by trasformations defined in the following way:

$$(a_1,a_2)\in P_{(\varepsilon,W)}(g)\longleftrightarrow H_{a_1}g\subset H_{a_2},\tag{5}$$

where $g\in G$, $a_1,a_2\in A$.

From results proved in [9] and [10] we can deduced the following properties of simplest representations.

Proposition 1. *Let (ε,W) be the determining pair of a semigroup $(G,\cdot)$. Then*

$$(g_1,g_2)\in\zeta_{(\varepsilon,W)}\longleftrightarrow(\forall x)(xg_1\notin W\longrightarrow xg_1\equiv xg_2(\varepsilon)),\tag{6}$$

$$(g_1,g_2)\in\xi_{(\varepsilon,W)}\longleftrightarrow(\forall x)(xg_1\notin W\wedge xg_2\notin W\longrightarrow xg_1\equiv xg_2(\varepsilon)),\tag{7}$$

$$(g_1,g_2)\in\delta_{(\varepsilon,W)}\longleftrightarrow(\forall x)(xg_1\notin W\longrightarrow xg_1g_2\notin W),\tag{8}$$

where $g_1,g_2\in G$, $x\in G^*$ and $\zeta_{(\varepsilon,W)}$, $\xi_{(\varepsilon,W)}$, $\delta_{(\varepsilon,W)}$ denote $\zeta_{P_{(\varepsilon,W)}}$, $\xi_{P_{(\varepsilon,W)}}$ and $\delta_{P_{(\varepsilon,W)}}$, respectively.

[1] $(G^*,\cdot)$ denotes a semigroup obtained from $(G,\cdot)$ by adjoining an identity $e\notin G$.

Proposition 2. *If a semigroup $(G, \cdot)$ and a semilattice $(G, \curlywedge)$ satisfy the identity*

$$x(y \curlywedge z) = xy \curlywedge xz, \tag{9}$$

then

$$P_{(\varepsilon,W)}(g_1 \curlywedge g_2) = P_{(\varepsilon,W)}(g_1) \cap P_{(\varepsilon,W)}(g_2) \tag{10}$$

holds for arbitrary elements $g_1, g_2 \in G$ and a determining pair (ε, W) of $(G, \cdot)$ if and only if

$$g_1 \in W \longrightarrow g_1 \curlywedge g_2 \in W, \tag{11}$$

$$g_1 \curlywedge g_2 \notin W \longrightarrow g_1 \equiv g_2(\varepsilon), \tag{12}$$

$$g_1 \notin W \wedge g_1 \equiv g_2(\varepsilon) \longrightarrow g_1 \curlywedge g_2 \equiv g_1(\varepsilon). \tag{13}$$

An analogous result was proved in [12] (see also [2]) for Menger algebras of rank n. For $n = 1$ it gives the above proposition.

4. Let $(G, \cdot, \curlywedge, \xi, \delta)$ be an algebraic system such that $(G, \cdot)$ is a semigroup, $(G, \curlywedge)$ – a semilattice, ξ – a left regular binary relation on $(G, \cdot)$ containing the natural order ζ of a semilattice $(G, \curlywedge)$, δ – a left ideal on $(G, \cdot)$. Assume that $(G, \cdot, \curlywedge, \xi, \delta)$ satisfies the condition (9), as well as the conditions:

$$x \leqslant y \ \wedge \ u \leqslant v \ \wedge \ y \downarrow v \longrightarrow u \downarrow x, \tag{14}$$

$$x \downarrow y \longrightarrow (x \curlywedge y)u = xu \curlywedge yu, \tag{15}$$

where $x, y, z, u, v \in G$, $x \leqslant y \longleftrightarrow (x, y) \in \zeta \longleftrightarrow x \curlywedge y = x$, $x \downarrow y \longleftrightarrow (x, y) \in \xi$, $x \vdash y \longleftrightarrow (x, y) \in \delta$. Moreover, we assume also that in a semigroup $(G^*, \cdot)$ with the adjoing identity e we have $e \leqslant e$, $e \vdash e$ and $x \vdash e$ for all $x \in G$.

Proposition 3. *If an algebraic system $(G, \cdot, \curlywedge, \xi, \delta)$ satisfies all the above conditions, then the relation ξ is reflexive and symmetric; the relation ζ is stable on a semigroup $(G, \cdot)$.*

Proof. The relation ξ is reflexive since $\zeta \subset \xi$ and ζ is a natural order of a semilattice $(G, \curlywedge)$. It also is symmetric because for any $(x, y) \in \xi$ we have $x \leqslant x$, $y \leqslant y$ and $x \downarrow y$, whence, by (14), we obtain $y \downarrow x$, i.e., $(y, x) \in \xi$.

To prove that ζ is stable on a semigroup $(G, \cdot)$ assume that $x \leqslant y$ holds for some $x, y \in G$. Then $x \curlywedge y = x$. Hence $z(x \curlywedge y) = zx$, which, by (9), gives $zx \curlywedge zy = zx$. Thus $zx \leqslant zy$. So, ζ is left regular. Since $\zeta \subset \xi$, from $x \leqslant y$ it follows $x \downarrow y$, which, by (15), implies $(x \curlywedge y)z = xz \curlywedge yz$. Hence $xz = xz \curlywedge yz$, i.e., $xz \leqslant yz$. This means that ζ is right regular. Consequently, ζ is stable on a semigroup $(G, \cdot)$. □

Definition 1. A subset $H \subset G$ is f_ξ-*closed* if the implication

$$u \downarrow v \ \wedge \ (u \curlywedge v)x \boxdot y \leqslant zt \ \wedge \ u, vx \in H \longrightarrow z \in H \tag{16}$$

is valid for all $x, y, t \in G^*$ and $z, u, v \in G$, where $x \boxdot y \leqslant z$ denotes the formula $x \vdash y \ \wedge \ xy \leqslant z$.

Clearly the set of all f_ξ-closed subsets of G forms a complete lattice of intersections of subsets, which determines the operation f_ξ. Obviously $f_\xi(X)$ is the least f_ξ-closed subset of G containing $X \subset G$.

Proposition 4. *A non-empty subset H of an algebraic system $(G, \cdot, \curlywedge, \xi, \delta)$ is f_ξ-closed if and only if it satisfies the conditions*

$$xy \in H \longrightarrow x \in H, \tag{17}$$
$$g_1 \vdash g_2 \,\wedge\, g_1 \in H \longrightarrow g_1 g_2 \in H, \tag{18}$$
$$g_1 \curlywedge g_2 = g_1 \in H \longrightarrow g_2 \in H, \tag{19}$$
$$g_1 \downarrow g_2 \,\wedge\, g_1, g_2 x \in H \longrightarrow (g_1 \curlywedge g_2)x \in H, \tag{20}$$

where x in the condition (20) *may be the empty symbol.*

Proof. Let H be a f_ξ-closed subset of G. Then

$$u \downarrow v \,\wedge\, (u \curlywedge v)x \vdash y \,\wedge\, (u \curlywedge v)xy \leqslant zt \,\wedge\, u, vx \in H \longrightarrow z \in H \tag{21}$$

for all $x, y, t \in G^*$ and $z, u, v \in G$.

Using (21) we can prove conditions (17) – (20). Indeed, for $u = v = xy$, $x = y = e$, $t = y$, $z = x$ the implication (21) has the form

$$xy \downarrow xy \,\wedge\, (xy \curlywedge xy)e \vdash e \,\wedge\, (xy \curlywedge xy)e \leqslant xy \,\wedge\, xy, xye \in H \longrightarrow x \in H.$$

Since relations ξ and ζ are reflexive and the operation $\curlywedge$ is idempotent, the last condition is equivalent to the implication (17).

For $u = v = g_1$, $x = e$, $y = g_1$, $t = e$, $z = g_1 g_2$ the implication (21) gives the condition

$$g_1 \downarrow g_1 \,\wedge\, (g_1 \curlywedge g_1)e \vdash g_2 \,\wedge\, (g_1 \curlywedge g_1)eg_2 \leqslant g_1 g_2 e \,\wedge\, g_1, g_1 e \in H \longrightarrow g_1 g_2 \in H,$$

which is equivalent to (18).

Similarly for $u = v = g_1$, $x = y = t = e$, $z = g_2$ from (21) we obtain

$$g_1 \downarrow g_1 \,\wedge\, (g_1 \curlywedge g_1)e \vdash e \,\wedge\, (g_1 \curlywedge g_1)ee \leqslant g_2 e \,\wedge\, g_1, g_1 e \in H \longrightarrow g_2 \in H,$$

i.e., $g_1 \leqslant g_2 \wedge g_1 \in H \longrightarrow g_2 \in H$. So, (21) implies (19).

Finally, (21) for $u = g_1$, $v = g_2$, $y = e$, $z = (g_1 \curlywedge g_2)x$, $t = e$, gives

$$g_1 \downarrow g_2 \wedge (g_1 \curlywedge g_2)x \vdash e \wedge (g_1 \curlywedge g_2)xe \leqslant (g_1 \curlywedge g_2)xe \wedge g_1, g_2 x \in H \longrightarrow (g_1 \curlywedge g_2)x \in H,$$

which implies (20).

To prove the converse assume that the conditions (17) (18), (19), (20) and the premise of (21) are satisfied. Then from $u \downarrow v \wedge u, vx \in H$, according to (20), we obtain $(u \curlywedge v)x \in H$. Since $(u \curlywedge v)x \vdash y$, by (18), the last condition implies $(u \curlywedge v)xy \in H$. But $(u \curlywedge v)xy \leqslant zt$, by (19), gives $zt \in H$, which by (17) gives $z \in H$. Thus, (17), (18), (19), (20) imply (21). $\square$

For a non-empty subset H of G we define the set

$$F_\xi(H) = \{z \mid (\exists u, v, x, y, t)(u \downarrow v \wedge (u \curlywedge v)x \boxdot y \leqslant zt \wedge u, vx \in H)\},$$

where $x, y, t \in G^*$ and $z, u, v \in G$.

Lemma 2. *For any subsets H, H_1, H_2 of G we have*
(*a*) $H \subset F_\xi(H)$,
(*b*) $F_\xi(H_1) \subset F_\xi(H_2)$ *for* $H_1 \subset H_2$.
(*c*) $F_\xi(H) = H$ *for any f_ξ-closed subset H of G.*

Proof. Indeed, if $z \in H$, then

$$z \downarrow z \wedge (z \curlywedge z)e \boxdot e \leqslant ze \wedge z, ze \in H,$$

which means that $z \in F_\xi(H)$. Hence, $H \subset F_\xi(H)$.

The second condition is obvious.

To prove the last condition assume that H is a f_ξ-closed subset of G. Then for any $z \in F_\xi(H)$ and some $x, y, t \in G^*$, $u, v \in G$ we have

$$u \downarrow v \wedge (u \curlywedge v)x \boxdot y \leqslant zt \wedge u, vx \in H.$$

Since H is f_ξ-closed, the above implies $z \in H$. Thus $F_\xi(H) \subset H$, which together with (a) proves $F_\xi(H) = H$. $\square$

Proposition 5. *For any subset H of an algebraic system $(G, \cdot, \curlywedge, \xi, \delta)$ holds*

$$f_\xi(H) = \bigcup_{n=0}^{\infty} \overset{n}{F}_\xi(H), \tag{22}$$

where $\overset{0}{F}_\xi(H) = H$ *and* $\overset{n}{F}_\xi(H) = F_\xi\Big(\overset{n-1}{F}_\xi(H)\Big)$ *for any positive integer n.*

Proof. By Lemma 2 we have $\overset{0}{F}_\xi(H) \subset \overset{1}{F}_\xi(H) \subset \overset{2}{F}_\xi(H) \subset \ldots$ for any subset H of G.

Let $\overline{H}_\xi = \bigcup_{n=0}^{\infty} \overset{n}{F}_\xi(H)$ and

$$u \downarrow v \wedge (u \curlywedge v)x \vdash y \wedge (u \curlywedge v)xy \leqslant zt \wedge u, vx \in \overline{H}_\xi\,,$$

for some $x, y, t \in G^*$ and $z, u, v \in G$. Since $u, vx \in \overline{H}_\xi$, there are natural numbers n_1, n_2 such that $u \in \overset{n_1}{F}_\xi(H)$ and $vx \in \overset{n_2}{F}_\xi(H)$. Hence $\overset{n_i}{F}_\xi(H) \subset \overset{n}{F}_\xi(H)$, $i = 1, 2$, for $n = \max(n_1, n_2)$. Therefore

$$u \downarrow v \wedge (u \curlywedge v)x \boxdot y \leqslant zt \wedge u, vx \in \overset{n}{F}_\xi(H)\,,$$

so, $z \in \overset{n+1}{F}_\xi(H) \subset \overline{H}_\xi$. This proves that $\overline{H}_\xi$ is a f_ξ-closed subset of G.

By the definition $H \subset f_\xi(H)$. Hence, by Lemma 2, $F_\xi(H) \subset F_\xi(f_\xi(H)) = f_\xi(H)$. Similarly, $\overset{2}{F}_\xi(H) \subset f_\xi(H)$, etc. Consequently, $\overset{n}{F}_\xi(H) \subset f_\xi(H)$ for any n, which implies $\bigcup\limits_{n=0}^{\infty} \overset{n}{F}_\xi(H) \subset f_\xi(H)$, i.e., $\overline{H}_\xi \subset f_\xi(H)$. On the other hand, $H \subset \bigcup\limits_{n=0}^{\infty} \overset{n}{F}_\xi(H) = \overline{H}_\xi$. Therefore $f_\xi(H) \subset f_\xi(\overline{H}_\xi) = \overline{H}_\xi$. Thus $\overline{H}_\xi = f_\xi(H)$, which completes the proof of (22). □

Using the method of mathematical induction we can easily prove the following proposition:

Proposition 6. *For each subset H of an algebraic system $(G, \cdot, \curlywedge, \xi, \delta)$, any natural number $n > 1$ and any $z \in G$ we have $z \in \overset{n}{F}_\xi(H)$ if and only if following system of conditions*

$$\left(\begin{array}{c} u_1 \downarrow v_1 \wedge (u_1 \curlywedge v_1)x_1 \boxdot y_1 \leqslant z t_1, \\ \bigwedge\limits_{i=1}^{2^{n-1}-1} \left(\begin{array}{l} u_{2i} \downarrow v_{2i} \wedge (u_{2i} \curlywedge v_{2i})x_{2i} \boxdot y_{2i} \leqslant u_i t_{2i}, \\ u_{2i+1} \downarrow v_{2i+1} \wedge (u_{2i+1} \curlywedge v_{2i+1})x_{2i+1} \boxdot y_{2i+1} \leqslant v_i x_i t_{2i+1} \end{array} \right), \\ \bigwedge\limits_{i=2^{n-1}}^{2^n-1} (u_i, v_i x_i \in H) \end{array} \right)$$

is valid for some $x_i, y_i, t_i \in G^*$ and $u_i, v_i \in G$.

In the sequel the system of the above conditions will be denoted by $\mathfrak{X}_n(z, H)$.

5. Let $(\Phi, \circ, \cap, \xi_\Phi, \delta_\Phi)$ be a transformative $\cap$-semigroup of transformations with the relation of semicompatibility ξ_Φ and the relation of semiadjacency δ_Φ.

Proposition 7. $\bigcap\limits_{\varphi_i \in H_\Phi} pr_1\, \varphi_i \subset pr_1\, \varphi$ *for every $H_\Phi \subset \Phi$ and $\varphi \in f_{\xi_\Phi}(H_\Phi)$.*

Proof. First we show that the following implication

$$\varphi \in \overset{n}{F}_{\xi_\Phi}(H_\Phi) \longrightarrow \bigcap_{\varphi_i \in H_\Phi} \mathrm{pr}_1\, \varphi_i \subset \mathrm{pr}_1\, \varphi \tag{23}$$

is valid for every integer n. We prove it by induction.

Let $\mathfrak{A} = \bigcap\limits_{\varphi_i \in H_\Phi} \mathrm{pr}_1\, \varphi_i$. If $n = 0$ and $\varphi \in \overset{0}{F}_{\xi_\Phi}(H_\Phi)$, then clearly $\varphi \in H_\Phi$. Thus $\mathfrak{A} \subset \mathrm{pr}_1\, \varphi$, which verifies (23) for $n = 0$.

Assume now that (23) is valid for some $n > 0$. To prove that it is valid for $n+1$ consider an arbitrary transformation $\varphi \in \overset{n+1}{F}_{\xi_\Phi}(H_\Phi)$. Then, for some transformations $x, y, t, u, v \in \Phi$, where x, y, t may be the empty symbols, we have $(u, v) \in \xi_\Phi$, $(x \circ (u \cap v), y) \in \delta_\Phi$, $y \circ x \circ (u \cap v) \subset t \circ \varphi$ and $u, x \circ v \in \overset{n}{F}_{\xi_\Phi}(H_\Phi)$. The last condition, according to the assumption on n, implies $\mathfrak{A} \subset \mathrm{pr}_1\, u$. Thus $\mathfrak{A} \subset \mathrm{pr}_1\, (x \circ v) \subset \mathrm{pr}_1\, v$. Consequently $\triangle_{\mathrm{pr}_1 u} \circ \triangle_{\mathfrak{A}} = \triangle_{\mathfrak{A}}$ and $\triangle_{\mathrm{pr}_1 v} \circ \triangle_{\mathfrak{A}} = \triangle_{\mathfrak{A}}$.

From $(x \circ (u \cap v), y) \in \delta_\Phi$ it follows $\mathrm{pr}_2\, (x \circ (u \cap v)) \subset \mathrm{pr}_1\, y$, which, by (2), gives $\mathrm{pr}_1\, (x \circ (u \cap v)) \subset \mathrm{pr}_1\, (y \circ x \circ (u \cap v)) \subset \mathrm{pr}_1\, (t \circ \varphi)$. Then, $(u, v) \in \xi_\Phi$

means that $u\circ\triangle_{\mathrm{pr}_1 v}=v\circ\triangle_{\mathrm{pr}_1 u}$. So, $u\circ\triangle_{\mathrm{pr}_1 v}\circ\triangle_{\mathfrak{A}}=v\circ\triangle_{\mathrm{pr}_1 u}\circ\triangle_{\mathfrak{A}}$, hence $u\circ\triangle_{\mathfrak{A}}=v\circ\triangle_{\mathfrak{A}}=u\circ\triangle_{\mathfrak{A}}\cap v\circ\triangle_{\mathfrak{A}}=(u\cap v)\circ\triangle_{\mathfrak{A}}$. Since $\mathfrak{A}\subset\mathrm{pr}_1\,(x\circ v)$, we have

$$
\begin{aligned}
\mathfrak{A}&\subset\mathrm{pr}_1\,(x\circ v\circ\triangle_{\mathfrak{A}})=\mathrm{pr}_1\,(x\circ(u\cap v)\circ\triangle_{\mathfrak{A}})\subset\mathrm{pr}_1\,(y\circ x\circ(u\cap v)\circ\triangle_{\mathfrak{A}})\\
&\subset\mathrm{pr}_1\,(t\circ\varphi\circ\triangle_{\mathfrak{A}})\subset\mathrm{pr}_1\,(\varphi\circ\triangle_{\mathfrak{A}})=\mathrm{pr}_1\,(\varphi\circ\triangle_{\mathrm{pr}_1\varphi}\circ\triangle_{\mathfrak{A}})\\
&=\mathrm{pr}_1\,(\varphi\circ\triangle_{\mathfrak{A}}\circ\triangle_{\mathrm{pr}_1\varphi})\subset\mathrm{pr}_1\,\varphi.
\end{aligned}
$$

Thus, $\mathfrak{A}\subset\mathrm{pr}_1\,\varphi$. This shows that (23) is valid for $n+1$. Consequently, (23) is valid for all integers n.

To complete the proof of this proposition observe now that, according to (22), for every $\varphi\in f_{\xi_\Phi}(H_\Phi)$ there exists n such that $\varphi\in\overset{n}{F}_{\xi_\Phi}(H_\Phi)$, which, by (23), gives $\bigcap\limits_{\varphi_i\in H_\Phi}\mathrm{pr}_1\,\varphi_i\subset\mathrm{pr}_1\,\varphi$. □

Theorem 1. *An algebraic system* $(G,\cdot,\curlywedge,\xi,\delta)$*, where* $(G,\cdot)$ *is a semigroup,* $(G,\curlywedge)$ *– a semilattice,* ξ,δ *— binary relations on* G*, is isomorphic to some transformative* $\cap$*-semigroup of transformations* $(\Phi,\circ,\cap,\xi_\Phi,\delta_\Phi)$ *if and only if* ξ *is a left regular relation containing a semilattice order* ζ*,* δ *is a left ideal of* $(G,\cdot)$*, the conditions* (9), (14), (15)*, as well as the conditions:*

$$x\curlywedge y\in f_\xi(\{x\})\longrightarrow x\leqslant y,\tag{24}$$

$$x\curlywedge y\in f_\xi(\{x,y\})\longrightarrow x\downarrow y,\tag{25}$$

$$xy\in f_\xi(\{x\})\longrightarrow x\vdash y\tag{26}$$

are satisfied by all elements of G*.*

Proof. Necessity. Let $(\Phi,\circ,\cap,\xi_\Phi,\delta_\Phi)$ be transformative $\cap$-semigroup of transformations of some set. We show that it satisfies all the conditions of our theorem.

The necessity of (9) is a consequence of results proved in [1] and [4]. Since the order ζ_Φ of a semilattice $(\Phi,\cap)$ coincides with the inclusion, ζ_Φ is contained in ξ_Φ. From (3) (Lemma 1) it follows that δ_Φ is a left ideal.

Let $(f,g)\in\xi_\Phi$, i.e., $f\circ\Delta_{\mathrm{pr}_1 g}=g\circ\Delta_{\mathrm{pr}_1 f}$. Then $f\circ\Delta_{\mathrm{pr}_1 g}\circ h=g\circ\Delta_{\mathrm{pr}_1 f}\circ h$. Since $\Delta_{\mathrm{pr}_1 g}\circ h=h\circ\Delta_{\mathrm{pr}_1 g\circ h}$ and $\Delta_{\mathrm{pr}_1 f}\circ h=h\circ\Delta_{\mathrm{pr}_1 f\circ h}$, we have $f\circ h\circ\Delta_{\mathrm{pr}_1 g\circ h}=g\circ h\circ\Delta_{\mathrm{pr}_1 f\circ h}$, which proves $(f\circ h,g\circ h)\in\xi_\Phi$. Thus, ξ_Φ is left regular.

If $f\subset g$, $h\subset p$ and $(g,p)\in\xi_\Phi$ for some $f,g,h,p\in\Phi$, then $f=g\circ\Delta_{\mathrm{pr}_1 f}$, $h=p\circ\Delta_{\mathrm{pr}_1 h}$ and $g\circ\Delta_{\mathrm{pr}_1 p}=p\circ\Delta_{\mathrm{pr}_1 g}$. The last equality implies $g\circ\Delta_{\mathrm{pr}_1 p}\circ\Delta_{\mathrm{pr}_1 f}\circ\Delta_{\mathrm{pr}_1 h}=p\circ\Delta_{\mathrm{pr}_1 g}\circ\Delta_{\mathrm{pr}_1 f}\circ\Delta_{\mathrm{pr}_1 h}$. Thus, $p\circ\Delta_{\mathrm{pr}_1 h}\circ\Delta_{\mathrm{pr}_1 g}\circ\Delta_{\mathrm{pr}_1 f}=g\circ\Delta_{\mathrm{pr}_1 f}\circ\Delta_{\mathrm{pr}_1 p}\circ\Delta_{\mathrm{pr}_1 h}$. Consequently, $h\circ\Delta_{\mathrm{pr}_1 g}\circ\Delta_{\mathrm{pr}_1 f}=f\circ\Delta_{\mathrm{pr}_1 p}\circ\Delta_{\mathrm{pr}_1 h}$, which in view of $\Delta_{\mathrm{pr}_1 g}\circ\Delta_{\mathrm{pr}_1 f}=\Delta_{\mathrm{pr}_1 f}$ and $\Delta_{\mathrm{pr}_1 p}\circ\Delta_{\mathrm{pr}_1 h}=\Delta_{\mathrm{pr}_1 h}$ gives $h\circ\Delta_{\mathrm{pr}_1 f}=f\circ\Delta_{\mathrm{pr}_1 h}$. Therefore, $(h,f)\in\xi_\Phi$. So, (14) is satisfied.

To prove (15) let $(f,g)\in\xi_\Phi$, i.e., $f\circ\Delta_{\mathrm{pr}_1 g}=g\circ\Delta_{\mathrm{pr}_1 f}$. Since $f\cap g=(f\cap g)\circ\Delta_{\mathrm{pr}_1 g}=f\circ\Delta_{\mathrm{pr}_1 g}\cap g=g\circ\Delta_{\mathrm{pr}_1 f}\cap g=g\circ\Delta_{\mathrm{pr}_1 f}\,(=f\circ\Delta_{\mathrm{pr}_1 g})$, we have $h\circ(f\cap g)=h\circ f\circ\Delta_{\mathrm{pr}_1 g}\cap h\circ g\circ\Delta_{\mathrm{pr}_1 f}=(h\circ f\cap h\circ g)\circ\Delta_{\mathrm{pr}_1 g}\circ\Delta_{\mathrm{pr}_1 f}=$

$h \circ f \circ \Delta_{\mathrm{pr}_1 f} \cap h \circ g \circ \Delta_{\mathrm{pr}_1 g} = h \circ f \cap h \circ g$. Thus $h \circ (f \cap g) = h \circ f \cap h \circ g$, which proves (15).

Now let $\varphi \cap \psi \in f_{\xi_\Phi}(\{\varphi\})$ for some $\varphi, \psi \in \Phi$. Then $\mathrm{pr}_1\,\varphi \subset \mathrm{pr}_1\,(\varphi \cap \psi)$, by Proposition 7. Hence $\mathrm{pr}_1\,(\varphi \cap \psi) = \mathrm{pr}_1\,\varphi$ since $\mathrm{pr}_1\,(\varphi \cap \psi) \subset \mathrm{pr}_1\,\varphi$. Thus $\varphi = \varphi \circ \triangle_{\mathrm{pr}_1\,\varphi} = \varphi \circ \triangle_{\mathrm{pr}_1\,(\varphi \cap \psi)} = \varphi \cap \psi \subset \psi$. This proves (24), because the inclusion $\subset$ coincides with the order ζ_Φ of the semilattice $(\Phi, \cap)$.

If $\varphi \cap \psi \in f_{\xi_\Phi}(\{\varphi, \psi\})$, then, by Proposition 7, $\mathrm{pr}_1\,\varphi \cap \mathrm{pr}_1\,\psi \subset \mathrm{pr}_1\,(\varphi \cap \psi)$, which together with the obvious inclusion $\mathrm{pr}_1\,(\varphi \cap \psi) \subset \mathrm{pr}_1\,\varphi \cap \mathrm{pr}_1\,\psi$ gives $\mathrm{pr}_1\,(\varphi \cap \psi) = \mathrm{pr}_1\,\varphi \cap \mathrm{pr}_1\,\psi$. So, $\varphi \circ \triangle_{\mathrm{pr}_1\,\psi} = \varphi \circ \triangle_{\mathrm{pr}_1\,\varphi} \circ \triangle_{\mathrm{pr}_1\,\psi} = \varphi \circ \triangle_{\mathrm{pr}_1\,\varphi \cap \mathrm{pr}_1\,\psi} = \varphi \circ \triangle_{\mathrm{pr}_1\,(\varphi \cap \psi)} = \varphi \cap \psi = \psi \circ \triangle_{\mathrm{pr}_1\,(\varphi \cap \psi)} = \psi \circ \triangle_{\mathrm{pr}_1\,\psi \cap \mathrm{pr}_1\,\varphi} = \psi \circ \triangle_{\mathrm{pr}_1\,\psi} \circ \triangle_{\mathrm{pr}_1\,\varphi} = \psi \circ \triangle_{\mathrm{pr}_1\,\varphi}$. Thus $\varphi \circ \triangle_{\mathrm{pr}_1\,\psi} = \psi \circ \triangle_{\mathrm{pr}_1\,\varphi}$, i.e., $(\varphi, \psi) \in \xi_\Phi$. This proves (25).

To prove the last condition let $\psi \circ \varphi \in f_{\xi_\Phi}(\{\varphi\})$. Then $\mathrm{pr}_1\,\varphi \subset \mathrm{pr}_1\,(\psi \circ \varphi)$, which by (2), gives $(\varphi, \psi) \in \delta_\Phi$. This means that (26) also is satisfied.

Sufficiency. Let $(G, \cdot, \curlywedge, \xi, \delta)$ be an algebraic system satisfying all the conditions of the theorem. Then, by Proposition 3, ξ is a reflexive and symmetric relation, and ζ is stable in a semigroup $(G, \cdot)$. Moreover, the implication

$$g_1 \leqslant g_2 \wedge g_1 \in f_\xi(\{x, y\}) \longrightarrow g_2 \in f_\xi(\{x, y\}) \tag{27}$$

is valid for all $g_1, g_2, x, y \in G$. In fact, the premise of (27) can be rewritten in the form:

$$g_1 \downarrow g_1 \wedge (g_1 \curlywedge g_1)e \boxdot e \leqslant g_2 e \wedge g_1, g_1 e \in f_\xi(\{x, y\}).$$

So, if it is satisfied, then, according to the definition of $F_\xi(H)$ and Lemma 2, $g_2 \in F_\xi(f_\xi(\{x, y\})) = f_\xi(\{x, y\})$, which proves (27).

Now we shows that for every $x, y \in G$ the subset $G \setminus f_\xi(\{x, y\})$ is a right ideal of a semigroup $(G, \cdot)$. Indeed, if $gu \in f_\xi(\{x, y\})$, then, by (22), for some natural n we have $gu \in \overset{n}{F}_\xi(\{x, y\})$. Hence

$$gu \downarrow gu \wedge (gu \curlywedge gu)e \boxdot e \leqslant gu \wedge gu, gue \in \overset{n}{F}_\xi(\{x, y\}),$$

so, $g \in \overset{n+1}{F}_\xi(\{x, y\}) \subset f_\xi(\{x, y\})$. Thus, $g \in f_\xi(\{x, y\})$. In this way we have shown the implication $gu \in f_\xi(\{x, y\}) \longrightarrow g \in f_\xi(\{x, y\})$, which by the contraposition is equivalent to the implication $g \notin f_\xi(\{x, y\}) \longrightarrow gu \notin f_\xi(\{x, y\})$. The last implication means that $G \setminus f_\xi(\{x, y\})$ is a right ideal.

If $u \downarrow v$ for $u, v \in f_\xi(\{x, y\})$, then, obviously,

$$u \downarrow v \wedge (u \curlywedge v)e \vdash e \wedge (u \curlywedge v)ee \leqslant (u \curlywedge v)e \wedge u, ve \in f_\xi(\{x, y\}).$$

Thus $u \curlywedge v \in F_\xi(f_\xi(\{x, y\})) = f_\xi(\{x, y\})$, since the set $f_\xi(\{x, y\})$ is f_ξ-closed. So, $f_\xi(\{x, y\})$ satisfies the implication

$$u \downarrow v \wedge u, v \in f_\xi(\{x, y\}) \longrightarrow u \curlywedge v \in f_\xi(\{x, y\}). \tag{28}$$

We show now that the relation

$$\varepsilon_{(g_1, g_2)} = \{(x, y) \,|\, x \curlywedge y \in f_\xi(\{g_1, g_2\}) \vee x, y \notin f_\xi(\{g_1, g_2\})\}$$

defined on a set G is a right regular equivalence and $G \setminus f_\xi(\{g_1, g_2\})$ is its equivalence class.

The reflexivity and symmetry of $\varepsilon_{(g_1,g_2)}$ are obvious. To prove the transitivity let $(x,y),(y,z) \in \varepsilon_{(g_1,g_2)}$. If $x,y,z \notin f_\xi(\{g_1,g_2\})$, then clearly $(x,z) \in \varepsilon_{(g_1,g_2)}$. In the case $x \curlywedge y \in f_\xi(\{g_1,g_2\})$ from $x \curlywedge y \leqslant y$, by (27), we conclude $y \in f_\xi(\{g_1,g_2\})$. Therefore $x, z \in f_\xi(\{g_1,g_2\})$. Consequently, $x \curlywedge y$, $y \curlywedge z \in f_\xi(\{g_1,g_2\})$. But $x \curlywedge y \leqslant y$, $y \curlywedge z \leqslant y$ and $y \downarrow y$, hence the last, by (14), implies $(x \curlywedge y) \downarrow (y \curlywedge z)$. From this, applying (28), we deduce $x \curlywedge y \curlywedge z \in f_\xi(\{g_1,g_2\})$. On the other hand $x \curlywedge y \curlywedge z \leqslant x \curlywedge z$ for all $x,y,z \in G$. So, $x \curlywedge y \curlywedge z \in f_\xi(\{g_1,g_2\})$, according to (27), implies $x \curlywedge z \in f_\xi(\{g_1,g_2\})$. Hence $(x,z) \in \varepsilon_{(g_1,g_2)}$. This proves the transitivity of $\varepsilon_{(g_1,g_2)}$. Summarizing $\varepsilon_{(g_1,g_2)}$ is an equivalence relation.

If $x,y \in G \setminus f_\xi(\{g_1,g_2\})$, then $(x,y) \in \varepsilon_{(g_1,g_2)}$. This means that a subset $G \setminus f_\xi(\{g_1,g_2\})$ is contained in some $\varepsilon_{(g_1,g_2)}$-class. Now let $x \in G \setminus f_\xi(\{g_1,g_2\})$ and $(x,y) \in \varepsilon_{(g_1,g_2)}$. The case $x \curlywedge y \in f_\xi(\{g_1,g_2\})$ is impossible, because in this case $x \in f_\xi(\{g_1,g_2\})$. So, $y \notin f_\xi(\{g_1,g_2\})$, i.e., $y \in G \setminus f_\xi(\{g_1,g_2\})$. Hence $G \setminus f_\xi(\{g_1,g_2\})$ coincides with some $\varepsilon_{(g_1,g_2)}$-class.

To prove that the relation $\varepsilon_{(g_1,g_2)}$ is right regular let $(x,y) \in \varepsilon_{(g_1,g_2)}$. If $x,y \in G \setminus f_\xi(\{g_1,g_2\})$, then $xz, yz \in G \setminus f_\xi(\{g_1,g_2\})$ since $G \setminus f_\xi(\{g_1,g_2\})$ is a right ideal. Thus $(xz,yz) \in f_\xi(\{g_1,g_2\})$. Now if $x \curlywedge y \in f_\xi(\{g_1,g_2\})$ and $xz \in f_\xi(\{g_1,g_2\})$. Then

$$(x \curlywedge y) \downarrow x \wedge (x \curlywedge y)z \vdash e \wedge (x \curlywedge y)ze \leqslant (x \curlywedge y)ze \wedge (x \curlywedge y), xz \in f_\xi(\{g_1,g_2\}),$$

whence, by (16), we obtain $(x \curlywedge y)z \in f_\xi(\{g_1,g_2\})$. But $(x \curlywedge y)z \leqslant yz$, hence $yz \in f_\xi(\{g_1,g_2\})$. Similarly, from $x \curlywedge y \in f_\xi(\{g_1,g_2\})$ and $yz \in f_\xi(\{g_1,g_2\})$ we get $xz \in f_\xi(\{g_1,g_2\})$. So, if $x \curlywedge y \in f_\xi(\{g_1,g_2\})$, then xz, yz belong or not belong to $f_\xi(\{g_1,g_2\})$ simultaneously. If $xz, yz \notin f_\xi(\{g_1,g_2\})$, then obviously, $(xz,yz) \in \varepsilon_{(g_1,g_2)}$. If $xz, yz \in f_\xi(\{g_1,g_2\})$, then, as it was shown above, from $x \curlywedge y \in f_\xi(\{g_1,g_2\})$ it follows $(x \curlywedge y)z \in f_\xi(\{g_1,g_2\})$. Since $(x \curlywedge y)z \leqslant xz$ and $(x \curlywedge y)z \leqslant yz$, then obviously $(x \curlywedge y)z \leqslant xz \curlywedge yz$. Hence $xz \curlywedge yz \in f_\xi(\{g_1,g_2\})$, i.e., $(xz,yz) \in \varepsilon_{(g_1,g_2)}$. So, in any case $(x,y) \in \varepsilon_{(g_1,g_2)}$ implies $(xz,yz) \in \varepsilon_{(g_1,g_2)}$. This proves that $\varepsilon_{(g_1,g_2)}$ is right regular.

With just shown it follows that the pair $(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})$, where

$$\varepsilon^*_{(g_1,g_2)} = \varepsilon_{(g_1,g_2)} \cup \{(e,e)\}, \qquad W_{(g_1,g_2)} = G \setminus f_\xi(\{g_1,g_2\}),$$

is the determining pair of a semigroup $(G,\cdot)$.

Let $\big(P_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}\big)_{(g_1,g_2)\in G\times G}$ be the family of simplest representations of a semigroup $(G,\cdot)$. Their sum

$$P = \sum_{(g_1,g_2)\in G\times G} P_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})} \tag{29}$$

is a representation of $(G,\cdot)$ by transformations. It is easy to see that the above determining pairs satisfy (11), (12) and (13). Therefore, by Proposition 2, we have

$$P_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}(x \curlywedge y) = P_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}(x) \cap P_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}(y)$$

for all $g_1, g_2 \in G$. Hence $P(x \curlywedge y) = P(x) \cap P(y)$ for $x, y \in G$. Thus, P is a homomorphism of an algebra $(G, \cdot, \curlywedge)$ onto a $\cap$-semigroup $(\Phi, \circ, \cap)$, where $\Phi = P(G)$.

Now we prove that $\xi = \xi_P$ and $\delta = \delta_P$. In fact, according to (4) and (7) we have

$$(x,y) \in \xi_P \quad \longleftrightarrow \bigcap_{(g_1,g_2)\in G\times G} \xi_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})} \ \longleftrightarrow$$

$$(\forall g_1)(\forall g_2)(\forall u \in G^*)\big(ux, uy \in f_\xi(\{g_1, g_2\}) \longrightarrow ux \curlywedge uy \in f_\xi(\{g_1, g_2\})\big).$$

The last implication for $u = e$ and $g_1 = x$, $g_2 = y$ has the form

$$x, y \in f_\xi(\{x, y\}) \longrightarrow x \curlywedge y \in f_\xi(\{x, y\}).$$

Thus $x \curlywedge y \in f_\xi(\{x, y\})$. Hence, by (25), we obtain $x \downarrow y$. This proves $\xi_P \subset \xi$.

To prove the converse inclusion let $(x, y) \in \xi$. If $ux, uy \in f_\xi(\{g_1, g_2\})$ for some $u \in G^*$ and $g_1, g_2 \in G$, then from $x \downarrow y$, by the left regularity of ξ, we obtain $ux \downarrow uy$, which by (28) implies $ux \curlywedge uy \in f_\xi(\{g_1, g_2\})$. Therefore $(ux, uy) \in \xi_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}$. Thus $(x, y) \in \bigcap\limits_{(g_1,g_2)\in G\times G} \xi_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})} = \xi_P$. So, $\xi \subset \xi_P$. Consequently, $\xi = \xi_P$.

Now if $(x, y) \in \delta$ and $ux \in f_\xi(\{g_1, g_2\})$ for some $g_1, g_2 \in G$ and $u \in G^*$, then also $(ux, y) \in \delta$ because δ is a left ideal of $(G, \cdot)$. Since $f_\xi(\{g_1, g_2\})$ is f_ξ-closed, $(ux, y) \in \delta$ together with $ux \in f_\xi(\{g_1, g_2\})$, according to (18), imply $uxy \in f_\xi(\{g_1, g_2\})$. Thus $(x, y) \in \delta_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}$. Hence $(x, y) \in \bigcap\limits_{(g_1,g_2)\in G\times G} \delta_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})} = \delta_P$. This proves $\delta \subset \delta_P$.

Conversely, let $(x, y) \in \delta_P$. Then, in view of (4) and (8), we have

$$(\forall g_1)\big(\forall g_2)(\forall u \in G^*)(ux \in f_\xi(\{g_1, g_2\}) \longrightarrow uxy \in f_\xi(\{g_1, g_2\})\big),$$

which for $u = e$ and $g_1 = g_2 = x$ has the form

$$x \in f_\xi(\{x\}) \longrightarrow xy \in f_\xi(\{x\}).$$

Thus $xy \in f_\xi(\{x\})$. This, by (26), implies $(x, y) \in \delta$. So, $\delta_P \subset \delta$, and consequently, $\delta_P = \delta$.

In this way we have shown that P is an homomorphism of $(G, \cdot, \curlywedge, \xi, \delta)$ onto a $\cap$-semigroup $(\Phi, \circ, \cap, \xi_\Phi, \delta_\Phi)$, where $\Phi = P(G)$.

It is an isomorphism. To prove this fact observe first that $\zeta_P \subset \zeta$. Indeed, according to (4) and (6), we have:

$$(x,y) \in \zeta_P \ \longleftrightarrow \bigcap_{(g_1,2)\in G\times G} \zeta_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)}))} \ \longleftrightarrow$$

$$(\forall g_1)(\forall g_2)(\forall u \in G^*)\big(ux \in f_\xi(g_1, g_2) \longrightarrow ux \curlywedge uy \in f_\xi(\{g_1, g_2\})\big).$$

Putting in the last implication $u = e$ and $g_1 = g_2 = x$ we obtain

$$x \in f_\xi(\{x\}) \longrightarrow x \curlywedge y \in f_\xi(\{x\}).$$

So, $x \curlywedge y \in f_\xi(\{x\})$. This, by (24), gives $x \leqslant y$, i.e., $(x,y) \in \zeta$. Hence $\zeta_P \subset \zeta$.

Now let $P(g_1) = P(g_2)$. Then $P(g_1) \subset P(g_2)$ and $P(g_2) \subset P(g_1)$. Hence $(g_1, g_2) \in \zeta_P$ and $(g_2, g_1) \in \zeta_P$. This implies $(g_1,g_2),(g_2,g_1) \in \zeta$. Thus $g_1 = g_2$ because ζ is a semilattice order. So, P is an isomorphism between $(G,\cdot,\curlywedge,\xi,\delta)$ and $(\Phi,\circ,\cap,\xi_\Phi,\delta_\Phi)$. $\square$

Now, using (22) and the formula $\mathfrak{X}_n(z,H)$ from our Proposition 6 we can write the conditions (24), (25) and (26) in the form of systems of elementary axioms $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$, respectively, where

$$
\begin{aligned}
A_n &\colon\ \mathfrak{X}_n(x \curlywedge y, \{x\}) \longrightarrow x \curlywedge y = x,\\
B_n &\colon\ \mathfrak{X}_n(x \curlywedge y, \{x,y\}) \longrightarrow (x,y) \in \xi,\\
C_n &\colon\ \mathfrak{X}_n(xy, \{x\}) \longrightarrow (x,y) \in \delta.
\end{aligned}
$$

Thus, we have proved the following theorem:

Theorem 2. *An algebraic system $(G,\cdot,\curlywedge,\xi,\delta)$, where $(G,\cdot)$ is a semigroup, $(G,\curlywedge)$ – a semilattice, ξ,δ — binary relations on G, is isomorphic to some transformative $\cap$-semigroup of transformations $(\Phi,\circ,\cap,\xi_\Phi,\delta_\Phi)$ if and only if ξ is a left regular relation containing a semilattice order ζ, δ is a left ideal of $(G,\cdot)$, the conditions* (9), (14), (15), *as well as the axioms systems $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ are satisfied by all elements of G.*

The relation of semicompatibility and the relation of semiadjacency in a semigroup of transformations can be characterized by essentially infinite systems of elementary axioms (for details see [9], [6] and [5]). Probably the axioms systems $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(C_n)_{n\in\mathbb{N}}$ are also essentially infinite, i.e., they are not equivalent to any its finite subsystems, but this problem requires further investigations.

References

[1] Clifford A. H., Preston G. B., *The algebraic theory of semigroups*, Amer. Math. Soc., Providence, R. I., vol. 1, 1964; vol. 2, 1967.

[2] Dudek W. A., Trokhimenko V. S., *Menger algebras of multiplace functions*, (Russian), Centrul Ed. USM, Chişinău 2006.

[3] Dudek W. A., Trokhimenko V. S., *The relation of semiadjacency of* $\cap$-*semigroups of transformations*, Semigroup Forum (in print).

[4] Garvackii V. S., $\cap$-*semigroups of transformations*, (Russian), Teor. Polugrupp Prilozh. **2** (1971), 3 – 13, (*Izdat. Saratov. Gos. Univ.*)

[5] Garvackiĭ V. S., Trokhimenko V. S., *A semiadjacence relation of certain transformation semigroups*, (Russian), Izv. Vysš. Učebn. Zaved. Matematika **2(117)** (1972), 23 – 32.

[6] Pavlovskiĭ E.V., *The relation of semiadjacence in semigroups of partial transformations*, (Russian), Izv. Vysš. Učebn. Zaved. Matematika **3(94)** (1970), 70 – 75.

[7] Saliĭ V. N., *A system axioms for transformative semigroups*, (Russian), Volzh. Mat. Sb. **5** (1966), 342 – 345.

[8] Saliĭ V. N., *Transformative semigroups* $(\mathfrak{G}, \circ, \xi, \chi_1, \chi_2)$, Sov. Math., Dokl. **9** (1968), 320 – 323 (translation from Dokl. Akad. Nauk SSSR **179** (1968), 30 – 33).

[9] Schein B. M., *Transformative semigroups of transformations*, Amer. Math. Soc., Translat., II. ser. **76** (1968), 1 – 19 (translation from Mat. Sb. **71** (1966), 65 – 82).

[10] Schein B. M., *Lectures on semigroups of transformations*, Amer. Math. Soc. Translat., II. ser. 113 (1979), 123 – 181.

[11] Trokhimenko V. S., *Semiadjacence relation in algebras of multiplace functions*, (Russian), Teor. Polugrupp Prilozh. **3** (1974) , 108 – 118 (*Izdat. Saratov. Gos. Univ.*).

[12] Trokhimenko V.S., *A characterization of* $\mathcal{P}$*-algebras of multiplace functions*, Siberian Math. J. **16** (1975), 461 – 470 (translation from Sibirsk. Mat. Zh. **16** (1975), 599 – 611).

[13] Trokhimenko V. S., *Some relativized* P*-algebras of multiplace functions*, Sov. Math. **24** (1980), 105 – 107 (translation from Izv. Vyssh. Uchebn. Zaved. Matematika **6** (1980), 85 – 86).

[14] Vagner V. V., *Transformative semigroups*, Amer. Math. Soc., Translat., II, ser. **36** (1964), 337 – 350 (translation from Izv. Vysš. Učebn. Zaved. Matematika **17** (1960), 36 – 48).

Dudek W. A.
Institute of Mathematics and
Computer Science
Wroclaw University of Technology
50-370 Wroclaw
Poland
Email: dudek@im.pwr.wroc.pl

Trokhimenko V. S.
Department of Mathematics
Pedagogical University
21100 Vinnitsa
Ukraine
Email: vtrokhim@gmail.com